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# An Eigenfunction Expansion for a Quadratic Pencil of a Schrödinger Operator with Spectral Singularities

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In this paper, we consider the operator $L$ generated in $L^2(\mathbf{R}_+)$ by the differential expression

$$\ell(y) = -y'' + [q(x) + 2\lambda p(x) - \lambda^2]\, y, \qquad x \in \mathbf{R}_+ = [0, \infty),$$

and the boundary condition $y(0) = 0$, where $p$ and $q$ are complex-valued functions and $p$ is continuously differentiable on $\mathbf{R}_+$. We derive a two-fold spectral expansion of $L$ (in the sense of Keldysh, 1951, *Soviet Math. Dokl.* **77**, 11–14 [1971, *Russian Math. Survey* **26**, 15–44 (Engl. transl.)]) in terms of the principal functions under the conditions

$$\lim_{x \to \infty} p(x) = 0, \qquad \sup_{x \in \mathbf{R}_+} \{e^{\varepsilon x}[\,|q(x)| + |p'(x)|\,]\} < \infty, \quad \varepsilon > 0,$$

taking into account the spectral singularities. Also we investigate the convergence of the spectral expansion. © 1999 Academic Press

## 1. INTRODUCTION

The spectral analysis of the non-selfadjoint abstract operators with purely discrete spectrum has been considered by Keldysh [10]. He studied the spectrum and principal functions (eigenfunctions and associated functions) of operators involving a polynomial dependent on the spectral parameter, and also showed the completeness of the principal functions of these operators in Hilbert space.

The spectral analysis of a non-selfadjoint differential operators with continuous and discrete spectrum was investigated by Naimark [24]. He



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showed the existence of spectral singularities in the continuous spectrum of the non-selfadjoint differential operator $L_0$, generated in $L^2(\mathbf{R}_+)$, by the differential expression

$$\ell_0(y) = -y'' + q(x)\,y, \qquad x \in \mathbf{R}_+ = [0, \infty), \tag{1.1}$$

with the boundary condition $y'(0) - hy(0) = 0$, where $q$ is a complex valued function, and $h \in \mathbf{C}$. Also, if $e^{\varepsilon x} q(x) \in L^1(\mathbf{R}_+)$ for some $\varepsilon > 0$, then $L_0$ has a finite number of eigenvalues and spectral singularities with finite multiplicities. Moreover, he derived the spectral expansion in some particular cases.

Another approach for the discussion of the spectral analysis of $L_0$ was given by Marchenko [22]. Let $E$ denote the set of all even entire functions of exponential type which are integrable over the real axis, and let $E'$ denote the dual of $E$. We define

$$\varphi(f_i, \lambda) = \int_0^\infty f_i(x)\, \varphi(x, \lambda)\, dx, \qquad i = 1, 2,$$

for any finite $f_1, f_2 \in L^2(\mathbf{R}_+)$, where $\varphi(x, \lambda)$ is the solution of $\ell_0(y) = \lambda^2 y$, subject to the initial conditions $\varphi(0, \lambda) = 1$, $\varphi_x(0, \lambda) = h$. In [22] Marchenko showed that

$$\varphi(f_1, \lambda),\ \varphi(f_2, \lambda) \in E,$$

and there exists a functional $T \in E'$ such that

$$\int_0^\infty f_1(x)\, f_2(x)\, dx = T[\varphi(f_1, \lambda) . \varphi(f_2, \lambda)]. \tag{1.2}$$

$T$ is the generalized spectral function of $L_0$. (1.2) is a generalization of the well-known Parseval equality for the singular selfadjoint differential operators, and is called Marchenko–Parseval equality.

The results of Naimark were extended to differential operators on the entire real axis by Kemp [11], and to the three dimensional Schrödinger operators by Gasymov [4].

Pavlov [26] established the dependence of the structure of the spectral singularities of $L_0$ on the behaviour of the potential function at infinity. He first gave the Marchenko–Parseval equality in the form of

$$\int_0^\infty f_1(x)\, f_2(x)\, dx = \frac{1}{2\pi i} \int_\gamma m(\lambda)\, \varphi(f_1, \lambda) . \varphi(f_2, \lambda)\, d\lambda, \tag{1.3}$$

where $\gamma$ is the contour enclosing the spectrum of $L_0$ and $m(\lambda)$ is the Weyl–Titchmarsh function [29]. Then he obtained the spectral expansion of $L_0$

in terms of principal functions, using (1.3) and the analytical properties of $m(\lambda)$.

In Gasymov's paper [3] the results of Naimark and Marchenko were extended to the case when the potential $q$ of $L_0$ is equal to $\ell(\ell+1)\,x^{-2}+V(x)$, where $V$ is a complex valued function, and

$$\int_0^\infty e^{\varepsilon x}\,|V(x)|\,dx<\infty,$$

for some $\varepsilon>0$.

One very important step in the spectral analysis of $L_0$ was taken by Lyance [18]. He showed that the spectral singularities play an important role in the spectral analysis of $L_0$. He also investigated the effect of the spectral singularities in the spectral expansion.

The eigenfunction expansion of the non-selfadjoint operator, generated in $L^2(\mathbf{R}_+)$ by (1.1) and the boundary condition

$$\int_0^\infty K(x)\,y(x)\,dx+\alpha y'(0)-\beta y(0)=0,$$

where $K\in L^2(\mathbf{R}_+)$ is a complex valued function, and $\alpha,\beta\in\mathbf{C}$, with $|\alpha|+|\beta|\neq 0$, was investigated in detail by Krall [12–16].

Note that the principal functions corresponding to the spectral singularities are not the elements of $L^2(\mathbf{R}_+)$ (i.e., these functions increase polynomially as $x\to\infty$ ). Also, the spectral singularities belong to the continuous spectrum and are the poles of the resolvent's kernel; but they are not eigenvalues. However, spectral singularities play a certain critical role in the spectral expansion. Their existence is accompanied by specific phenomenon which are new in the sense that they do not occur either in the spectral theory of selfadjoint or normal operators.

Lyance [18] studied the role of spectral singularities in the spectal expansion with respect to principal functions for the Sturm–Liouville operator $L_0$ by use of Fourier $L_0$-transforms. In order to utilize this technique, the potential of $L_0$ should satisfy Naimark condition

$$\int_0^\infty e^{\varepsilon x}\,|q(x)|\,dx<\infty,\qquad \varepsilon>0.$$

The Laurent expansion of the resolvents of the abstract non-selfadjoint operators in neighborhood of spectral singularities was investigated by Gasymov–Maksudov [5] and Maksudov–Allakhverdiev [20]. They also studied the effect of spectral singularities in the spectral analysis of these operators.

The spectral analysis of some classes of dissipative operators with spectral singularities was considered by Pavlov [27] using the theory of functional models [23] and scattering theory [17].

Some problems of spectral theory of differential and some others types of operators with spectral singularities were also studied in [7, 25, 28, 30].

Let us consider the quadratic pencil of the Schrödinger operator $L$ generated in $L^2(\mathbf{R}_+)$ by the differential expression

$$\ell(y) = -y'' + [q(x) + 2\lambda p(x) - \lambda^2]\, y, \qquad x \in \mathbf{R}_+,$$

and the boundary condition $y(0) = 0$, where $p$ and $q$ are complex valued functions and $p$ is continuously differentiable on $\mathbf{R}_+$.

Let $\tilde{L}$ denote the operator generated in $L^2(\mathbf{R}_+)$ by the differential expression

$$\tilde{\ell}(y) = -y'' - [\lambda - p(x)]^2\, y, \qquad x \in \mathbf{R}_+,$$

and the boundary condition $y(0) = 0$. Note that in relativistic quantum mechanics the equation

$$y'' + [\lambda - p(x)]^2\, y = 0, \qquad x \in \mathbf{R}_+,$$

is called the Klein–Gordon $s$-wave equation for a particle of zero mass with static potential $p$ [6]. The operators $L_0$ and $\tilde{L}$ are particular case of $L$.

Some problems of spectral theory of $L$ and $\tilde{L}$ were studied by Degasperis [2], Jaulent–Jean [8, 9], Maksudov [19] and Maksudov–Guseinov [21].

In this paper, we derive a two-fold spectral expansion of $L$ in terms of the principal functions, under the conditions

$$\lim_{x \to \infty} p(x) = 0, \qquad \sup_{x \in \mathbf{R}_+} \{e^{\varepsilon x}[\,|q(x)| + |p'(x)|\,]\} < \infty, \qquad \varepsilon > 0, \tag{1.4}$$

taking into account the spectral singularities. Moreover the convergence of the spectral expansion is investigated. Note that we investigate the effect of the spectral singularities in the spectral expansion of $L$ by the regularization of divergent integrals using summability factors.

## 2. SPECIAL SOLUTIONS OF $\ell(y) = 0$.

Related with the operator $L$ we will consider the boundary value problem

$$-y'' + [q(x) + 2\lambda p(x) - \lambda^2]\, y = 0, \qquad x \in \mathbf{R}_+, \tag{2.1}$$

$$y(0) = 0, \tag{2.2}$$

assuming that the condition (1.4) holds.

Under the condition (1.4) the equation (2.1) has the solutions

$$e^+(x, \lambda) = e^{iw(x) + i\lambda x} + \int_x^\infty A^+(x, t)\, e^{i\lambda t}\, dt, \tag{2.3}$$

and

$$e^{-}(x, \lambda) = e^{-iw(x) - i\lambda x} + \int_x^\infty A^{-}(x, t)\, e^{-i\lambda t}\, dt, \tag{2.4}$$

for $\lambda \in \overline{\mathbf{C}}_{+} = \{\lambda: \lambda \in \mathbf{C}, \operatorname{Im} \lambda \geqslant 0\}$, and $\lambda \in \overline{\mathbf{C}}_{-} = \{\lambda: \lambda \in \mathbf{C}, \operatorname{Im} \lambda \leqslant 0\}$, respectively, where $w(x) = \int_x^\infty p(t)\, dt$, and the kernels $A^{\pm}(x, t)$ are expressed in terms of $p$ and $q$, and

$$|A^{\pm}(x, t)| \leqslant c\xi\left(\frac{x+t}{2}\right) \exp\{\zeta(x)\}, \tag{2.5}$$

where

$$\xi(x) = \int_x^\infty [\,|q(t)| + |p'(t)|\,]\, dt, \qquad \zeta(x) = \int_x^\infty [\,t\,|q(t)| + 2\,|p(t)|\,]\, dt, \tag{2.6}$$

and $c > 0$ is a constant. Therefore $e^{+}(x, \lambda)$ and $e^{-}(x, \lambda)$ are analytic with respect to $\lambda$ in $\mathbf{C}_{+} = \{\lambda: \lambda \in \mathbf{C}, \operatorname{Im} \lambda > 0\}$ and $\mathbf{C}_{-} = \{\lambda: \lambda \in \mathbf{C}, \operatorname{Im} \lambda < 0\}$, respectively, and continuous up to the real axis. $e^{\pm}(x, \lambda)$ also satisfy

$$e^{\pm}(x, \lambda) = e^{\pm i[w(x) + \lambda x]} + O\left(\frac{e^{\mp x \operatorname{Im} \lambda}}{|\lambda|}\right), \qquad \lambda \in \overline{\mathbf{C}}_{\pm}, \quad |\lambda| \to \infty, \tag{2.7}$$

$$e_x^{\pm}(x, \lambda) = \pm i[\lambda - p(x)]\, e^{\pm i[w(x) + \lambda x]} + O(1), \qquad \lambda \in \overline{\mathbf{C}}_{\pm}, \quad |\lambda| \to \infty. \tag{2.8}$$

Let $\varphi(x, \lambda)$ be the solution of (2.1) subject to the initial conditions $\varphi(0, \lambda) = 0$, $\varphi_x(0, \lambda) = 1$. $\varphi(x, \lambda)$ is an entire function of $\lambda$ and satisfies

$$\varphi(x, \lambda) = \frac{1}{\lambda} \sin\left\{\lambda x - \int_0^\infty p(t)\, dt\right\} + O\left(\frac{e^{x \operatorname{Im} \lambda}}{|\lambda|^2}\right), \qquad |\lambda| \to \infty. \tag{2.9}$$

The results stated above were obtained by Jaulent–Jean [8].

## 3. THE SPECTRUM OF $L$

By $\sigma_c(L), \sigma_d(L)$ and $\sigma_{ss}(L)$ we denote the continuous spectrum, the eigenvalues and the spectal singularities of $L$, respectively. We have previously shown [1] that

$$\sigma_c(L) = (-\infty, \infty),$$

$$\sigma_d(L) = \{\lambda: \lambda \in \mathbf{C}_{+}, e^{+}(\lambda) = 0\} \cup \{\lambda: \lambda \in \mathbf{C}_{-}, e^{-}(\lambda) = 0\},$$

$$\sigma_{ss}(L) = \{\lambda: \lambda \in \mathbf{R}^{*}, e^{+}(\lambda) = 0\} \cup \{\lambda: \lambda \in \mathbf{R}^{*}, e^{-}(\lambda) = 0\},$$

where

$$e^{\pm}(\lambda)=e^{\pm}(0,\lambda), \qquad \mathbf{R}^*=\mathbf{R}\setminus\{0\}.$$

Under the condition (1.4), we know that $L$ has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity [1]. Let $\lambda_1^+, ..., \lambda_j^+$ and $\lambda_1^-, ..., \lambda_k^-$ denote the zeros of the functions $e^+$ in $\mathbf{C}_+$ and $e^-$ in $\mathbf{C}_-$ (which are the eigenvalues of $L$) with multiplicities $m_1^+, ..., m_j^+$ and $m_1^-, ..., m_k^-$, respectively. Similarly, let $\lambda_1, ..., \lambda_v$ and $\lambda_{v+1}, ..., \lambda_\ell$ be zeros of $e^+$ and $e^-$ in $\mathbf{R}^*$ (the spectral singularities of $L$) with multiplicities $n_1, ..., n_v$ and $n_{v+1}, ..., n_\ell$, respectively.

We will also need the Hilbert spaces

$$H_n=\left\{f: \int_0^\infty (1+x)^{2n} |f(x)|^2\, dx<\infty\right\}, \qquad n=1, 2, ...,$$

$$H_{-n}=\left\{g: \int_0^\infty (1+x)^{-2n} |g(x)|^2\, dx<\infty\right\}, \qquad n=1, 2, ...,$$

with

$$\|f\|_n^2=\int_0^\infty (1+x)^{2n} |f(x)|^2\, dx, \qquad \|g\|_{-n}^2=\int_0^\infty (1+x)^{-2n} |g(x)|^2\, dx,$$

respectively. It is clear that,

$$H_{(n+1)} \subsetneqq H_n \subsetneqq L^2(\mathbf{R}_+) \subsetneqq H_{-n} \subsetneqq H_{-(n+1)}, \qquad n=1, 2, ...,$$

and $H_{-n}$ is isomorphic to the dual of $H_n$: $H_n' \sim H_{-n}$. In fact, for every functional $F\in H_n'$, there is a function $f^*$ belonging to $H_{-n}$ such that

$$F(f)=\int_0^\infty f(x)\, f^*(x)\, dx$$

for all $f\in H_n$.

We have previously shown that [1]:

$$\left.\begin{array}{lll} \left\{\dfrac{\partial^n}{\partial\lambda^n}\varphi(\cdot,\lambda)\right\}_{\lambda=\lambda_i^+}\in L^2(\mathbf{R}_+), & n=0, 1, ..., m_i^+-1, & i=1, ..., j, \\ \left\{\dfrac{\partial^n}{\partial\lambda^n}\varphi(\cdot,\lambda)\right\}_{\lambda=\lambda_i^-}\in L^2(\mathbf{R}_+), & n=0, 1, ..., m_i^--1, & i=1, ..., k, \end{array}\right\} \tag{3.1}$$

$$\left\{\frac{\partial^n}{\partial\lambda^n}\varphi(\cdot,\lambda)\right\}_{\lambda=\lambda_i}\in H_{-(n+1)}, \qquad n=0, 1, ..., n_i-1, \qquad i=1, ..., v, v+1, ..., \ell. \tag{3.2}$$

LEMMA 3.1. *For* $\lambda \in \mathbf{R}^*$,

$$\frac{\partial^n}{\partial \lambda^n} \varphi(\cdot, \lambda) \in H_{-(n+1)}. \tag{3.3}$$

*Proof.* It is known that, for $\lambda \in \mathbf{R}^*$,

$$2i\lambda\varphi(x, \lambda) = e^-(\lambda)\, e^+(x, \lambda) - e^+(\lambda)\, e^-(x, \lambda). \tag{3.4}$$

Then the proof of the lemma is the direct consequence of (2.3), (2.4), and (3.4). ∎

Let

$$G(x, t; \lambda) = \begin{cases} G^+(x, t; \lambda), & \lambda \in \mathbf{C}_+, \\ G^-(x, t; \lambda), & \lambda \in \mathbf{C}_-, \end{cases} \tag{3.5}$$

be the Green function of $L$, where

$$G^\pm(x, t; \lambda) = \frac{1}{e^\pm(\lambda)} \begin{cases} e^\pm(x, \lambda)\, \varphi(t, \lambda), & 0 \leqslant t < x, \\ e^\pm(t, \lambda)\, \varphi(x, \lambda), & x \leqslant t < \infty. \end{cases} \tag{3.6}$$

## 4. THE SPECTRAL EXPANSION

Let $C_0^\infty(\mathbf{R}_+)$ denote the set of infinitely differentiable functions in $\mathbf{R}_+$ with compact support. Then

$$f(x) = \int_0^\infty G(x, t; \lambda)[-f''(t) + q(t)\, f(t) + 2\lambda p(t)\, f(t) - \lambda^2 f(t)]\, dt,$$

for each $f \in C_0^\infty(\mathbf{R}_+)$. Hence we have

$$\frac{f(x)}{\lambda} = \frac{1}{\lambda} \int_0^\infty G(x, t; \lambda)\, \Theta(t)\, dt + 2 \int_0^\infty G(x, t; \lambda)\, p(t)\, f(t)\, dt - \lambda D(x, \lambda), \tag{4.1}$$

where

$$\Theta(t) = -f''(t) + q(t)\, f(t), \qquad D(x, \lambda) = \int_0^\infty G(x, t; \lambda)\, f(t)\, dt.$$

Let $\gamma_r$ denote the disc with center at the origin having radius $r$; let $\partial\gamma_r$ be the boundary of $\gamma_r$. $r$ will be chosen so that all eigenvalues and spectral singularities of $L$ are in $\gamma_r$. $P_{r\eta}$ denotes the part of $\gamma_r$ lying in the strip $|\mathrm{Im}\, \lambda| \leqslant \eta$ and $\gamma_{r\eta} = \gamma_{r\eta}^+ \cup \gamma_{r\eta}^-$, where $\gamma_{r\eta}^+$ and $\gamma_{r\eta}^-$ are the parts of $\gamma_r \setminus P_{r\eta}$ in the upper and lower half-planes, respectively (see Fig. 1). We choose $\eta$ so small that $P_{r\eta}$ does not contain any eigenvalues of $L$.

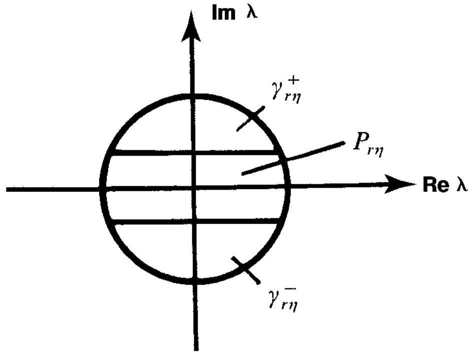


**FIGURE 1**

As it is seen in Fig.1,

$$\partial\gamma_r = \partial\gamma_{r\eta} \cup \partial P_{r\eta}. \tag{4.2}$$

From (4.1) we obtain

$$\begin{aligned} f(x) = &\frac{1}{2\pi i}\int_{\partial\gamma_r}\left\{\frac{1}{\lambda}\int_0^\infty G(x, t; \lambda)\, \Theta(t)\, dt\right\} d\lambda \\ &+\frac{1}{\pi i}\int_{\partial\gamma_r}\left\{\int_0^\infty G(x, t; \lambda)\, p(t)\, f(t)\, dt\right\} d\lambda \\ &-\frac{1}{2\pi i}\int_{\partial\gamma_r} \lambda D(x, \lambda)\, d\lambda. \end{aligned} \tag{4.3}$$

Using (2.7), (2.9), (3.5), (3.6) and Jordan's lemma, we see that the first term of the right-hand side of (4.3) vanishes as $r \to \infty$. The same result holds for the second term. This can be obtained from (2.7)–(2.9) utilizing integration by parts. Then considering (4.2) we find

$$f(x) = -\lim_{\substack{r\to\infty \\ \eta\to 0}} \frac{1}{2\pi i}\int_{\partial\gamma_{r\eta}} \lambda D(x, \lambda)\, d\lambda - \lim_{\substack{r\to\infty \\ \eta\to 0}} \frac{1}{2\pi i}\int_{\partial P_{r\eta}} \lambda D(x, \lambda)\, d\lambda. \tag{4.4}$$

Therefore

$$\frac{1}{2\pi i}\int_{\partial\gamma_{r\eta}} \lambda D(x, \lambda)\, d\lambda = \sum_{i=1}^{j} \operatorname*{Res}_{\lambda=\lambda_i^+} [\lambda D^+(x, \lambda)] + \sum_{i=1}^{k} \operatorname*{Res}_{\lambda=\lambda_i^-} [\lambda D^-(x, \lambda)],$$

where

$$D^\pm(x, \lambda) = \int_0^\infty G^\pm(x, t; \lambda)\, f(t)\, dt.$$

Let $\Gamma_+$ be the contour which isolates the real zeros of $e^+$ by semicircles with centers at $\lambda_i$, $i=1, ..., v$, having the same radius $\delta_0$ in the upper half-plane. Similarly $\Gamma_-$ will denote the corresponding contour for the real zeros of $e^-$ in the lower half-plane. The radius of semicircles being chosen so small that their diameters are mutually disjoint and do not contain the point $\lambda=0$ (see Fig. 2).

As it is easily seen from Fig. 1, we find

$$\lim_{\substack{r\to\infty\\ \eta\to 0}} \frac{1}{2\pi i}\int_{P\gamma_{r\eta}} \lambda D(x,\lambda)\, d\lambda = \frac{1}{2\pi i}\int_{\Gamma_-} \lambda D^-(x,\lambda)\, d\lambda - \frac{1}{2\pi i}\int_{\Gamma_+} \lambda D^+(x,\lambda)\, d\lambda$$

So (4.4) can be written as

$$f(x) = -\sum_{i=1}^{j} \operatorname*{Res}_{\lambda=\lambda_i^+} [\lambda D^+(x,\lambda)] - \sum_{i=1}^{k} \operatorname*{Res}_{\lambda=\lambda_i^-} [\lambda D^-(x,\lambda)] + \frac{1}{2\pi i}\int_{\Gamma_+} \lambda D^+(x,\lambda)\, d\lambda - \frac{1}{2\pi i}\int_{\Gamma_-} \lambda D^-(x,\lambda)\, d\lambda. \tag{4.5}$$

LEMMA 4.1. *For every $f \in C_0^\infty(\mathbf{R}_+)$*

$$\begin{aligned} f(x) = {} & \sum_{i=1}^{j} \left\{ \left(\frac{\partial}{\partial\lambda}\right)^{m_i^+ -1} [a_i^+(\lambda)\, \varphi(x,\lambda)\, \varphi(f,\lambda)] \right\}_{\lambda=\lambda_i^+} \\ & + \sum_{i=1}^{k} \left\{ \left(\frac{\partial}{\partial\lambda}\right)^{m_i^- -1} [a_i^-(\lambda)\, \varphi(x,\lambda)\, \varphi(f,\lambda)] \right\}_{\lambda=\lambda_i^-} \\ & + \frac{1}{2\pi i}\int_{\Gamma_+} \frac{\lambda e_x^+(0,\lambda)}{e^+(\lambda)}\, \varphi(x,\lambda)\, \varphi(f,\lambda)\, d\lambda \\ & - \frac{1}{2\pi i}\int_{\Gamma_-} \frac{\lambda e_x^-(0,\lambda)}{e^-(\lambda)}\, \varphi(x,\lambda)\, \varphi(f,\lambda)\, d\lambda, \end{aligned} \tag{4.6}$$

$$\begin{aligned} 0 = {} & \sum_{i=1}^{j} \left\{ \left(\frac{\partial}{\partial\lambda}\right)^{m_i^+ -1} [b_i^+(\lambda)\, \varphi(x,\lambda)\, \varphi(f,\lambda)] \right\}_{\lambda=\lambda_i^+} \\ & + \sum_{i=1}^{k} \left\{ \left(\frac{\partial}{\partial\lambda}\right)^{m_i^- -1} [b_i^-(\lambda)\, \varphi(x,\lambda)\, \varphi(f,\lambda)] \right\}_{\lambda=\lambda_i^-} \\ & + \frac{1}{2\pi i}\int_{\Gamma_+} \frac{e_x^+(0,\lambda)}{e^+(\lambda)}\, \varphi(x,\lambda)\, \varphi(f,\lambda)\, d\lambda \\ & - \frac{1}{2\pi i}\int_{\Gamma_-} \frac{e_x^-(0,\lambda)}{e^-(\lambda)}\, \varphi(x,\lambda)\, \varphi(f,\lambda)\, d\lambda, \end{aligned} \tag{4.7}$$

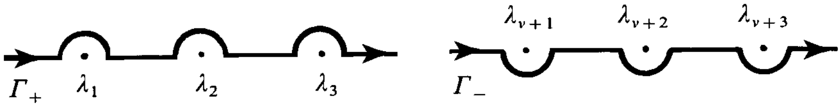


**FIGURE 2**

where

$$\left.\begin{aligned} a_i^+(\lambda) &= -\frac{\lambda(\lambda-\lambda_i^+)^{m_i^+}\, e_x^+(0,\lambda)}{(m_i^+-1)!\, e^+(\lambda)}, & i&=1,\ldots,j, \\ a_i^-(\lambda) &= -\frac{\lambda(\lambda-\lambda_i^-)^{m_i^-}\, e_x^-(0,\lambda)}{(m_i^--1)!\, e^-(\lambda)}, & i&=1,\ldots,k, \end{aligned}\right\} \tag{4.8}$$

$$\left.\begin{aligned} b_i^+(\lambda) &= -\frac{(\lambda-\lambda_i^+)^{m_i^+}\, e_x^+(0,\lambda)}{(m_i^+-1)!\, e^+(\lambda)}, & i&=1,\ldots,j, \\ b_i^-(\lambda) &= -\frac{(\lambda-\lambda_i^-)^{m_i^-}\, e_x^-(0,\lambda)}{(m_i^--1)!\, e^-(\lambda)}, & i&=1,\ldots,k, \end{aligned}\right\} \tag{4.9}$$

and

$$\varphi(f,\lambda) = \int_0^\infty f(t)\,\varphi(t,\lambda)\,dt.$$

*Proof.* Let $\psi(x,\lambda)$ be the solution of (2.1) subject to the initial conditions $\psi(0,\lambda)=1$, $\psi_x(0,\lambda)=0$. Then

$$G^\pm(x,t;\lambda) = \frac{e_x^\pm(0,\lambda)}{e^\pm(\lambda)}\,\varphi(x,\lambda)\,\varphi(t,\lambda) + a(x,t;\lambda), \tag{4.10}$$

where

$$a(x,t;\lambda) = \begin{cases} \psi(x,\lambda)\,\varphi(t,\lambda), & 0 \leqslant t < x, \\ \psi(t,\lambda)\,\varphi(x,\lambda), & x \leqslant t < \infty, \end{cases}$$

and $a(x,t;\lambda)$ is an entire function of $\lambda$. From (4.5) and (4.10) we obtain (4.6). Writing (4.1) as

$$\frac{f(x)}{\lambda^2} = \frac{1}{\lambda^2}\int_0^\infty G(x,t;\lambda)\,\Theta(t)\,dt + \frac{2}{\lambda}\int_0^\infty G(x,t,\lambda)\,p(t)\,f(t)\,dt - D(x,\lambda),$$

and repeating the calculation as we have done for (4.1), we have (4.7). ∎

Since the contours $\Gamma_+$ and $\Gamma_-$ in (4.6) and (4.7) do not coincide with the continuous spectrum of $L$, these formulae contains "non-spectral" objects. The purpose of this article is to transform (4.6) and (4.7) a into two-fold spectral expansion with respect to the principal functions of $L$.

LEMMA 4.2. *For any $f \in C_0^\infty(\mathbf{R}_+)$ there exists a constant $c > 0$ so that*

$$\int_{-\infty}^{\infty} |\lambda \varphi(f, \lambda)|^2 \, d\lambda \leqslant c \int_0^\infty |f(x)|^2 \, dx. \tag{4.11}$$

*Proof.* From (3.4) we get

$$2i\lambda \varphi(f, \lambda) = e^-(\lambda)\, e^+(f, \lambda) - e^+(\lambda)\, e^-(f, \lambda), \qquad \lambda \in \mathbf{R}^*, \tag{4.12}$$

where

$$e^\pm(f, \lambda) = \int_0^\infty f(x)\, e^\pm(x, \lambda) \, dx.$$

Using (2.3) we have

$$e^+(f, \lambda) = \int_0^\infty [(\mathbf{A}_0^+ + \mathbf{A}^+) f(t)] \, e^{i\lambda t} \, dt, \tag{4.13}$$

in which the operators $\mathbf{A}_0^+$ and $\mathbf{A}^+$ are defined by

$$\mathbf{A}_0^+ f(t) = e^{iw(t)} f(t), \qquad \mathbf{A}^+ f(t) = \int_0^t A^+(x, t) f(x) \, dx.$$

From (1.4) and (2.5) we find

$$|A^+(x, t)| \leqslant c_0 \exp\left\{-\frac{\varepsilon}{4}(x + t)\right\},$$

where $c_0 > 0$ is a constant. Hence $\mathbf{A}^+$ is a compact operator in $L^2(\mathbf{R}_+)$. Thus $(\mathbf{A}_0^+ + \mathbf{A}^+)$ is continuous and one-to-one on $L^2(\mathbf{R}_+)$. Using the Parseval's equality for the Fourier transforms and (4.13) we get

$$\int_{-\infty}^{\infty} |e^+(f, \lambda)|^2 \, d\lambda \leqslant c_1 \int_0^\infty |f(x)|^2 \, dx, \tag{4.14}$$

where $c_1 > 0$ is a constant. In a similar way, we also find

$$\int_{-\infty}^{\infty} |e^-(f, \lambda)|^2 \, d\lambda \leqslant c_2 \int_0^\infty |f(x)|^2 \, dx. \tag{4.15}$$

The proof of the lemma is completed by (2.7), (4.12), (4.14) and (4.15). ∎

By the preceding lemma for every function $f \in L^2(\mathbf{R}_+)$ the limit

$$\varphi(f, \lambda) = \lim_{N \to \infty} \int_0^N f(x)\, \varphi(x, \lambda)\, dx$$

exists in the sense of convergence in the mean square, relative to the measure $\lambda^2\, d\lambda$ on the real axis; that is,

$$\lim_{N \to \infty} \int_{-\infty}^{\infty} \left| \varphi(f, \lambda) - \int_0^N f(x)\, \varphi(x, \lambda)\, dx \right|^2 \lambda^2\, d\lambda = 0. \tag{4.16}$$

Since $C_0^\infty(\mathbf{R}_+)$ is dense in $L^2(\mathbf{R}_+)$, the estimate (4.11) may be extended onto $L^2(\mathbf{R}_+)$ for any $f \in L^2(\mathbf{R}_+)$ as

$$\int_{-\infty}^{\infty} |\lambda \varphi(f, \lambda)|^2\, d\lambda \leqslant c \int_0^\infty |f(x)|^2\, dx, \tag{4.17}$$

where $\varphi(f, \lambda)$ must be understood in the sense of (4.16). We shall need a generalization of this estimate.

LEMMA 4.3. *If $f \in H_m$, then $\varphi(f, \lambda)$ has a derivative of order $(m-1)$ which is absolutely continuous of every finite subinterval of the real axis and satisfies*

$$\int_{-\infty}^{\infty} \left| \left( \frac{d}{d\lambda} \right)^n [\lambda \varphi(f, \lambda)] \right|^2 d\lambda \leqslant c_n \int_0^\infty (1+x)^{2n}\, |f(x)|^2\, dx, \qquad n = 1, ..., m, \tag{4.18}$$

*where $c_n > 0$ are constants, $n = 1, ..., m$.*

The proof is similar to that of Lemma 4.2.

In order to transform (4.6) and (4.7) into the spectral expansion of $L$, we have to reform the integrals over $\Gamma_+$ and $\Gamma_-$ onto the real axis (i.e., to the continuous spectrum of $L$).

Since the spectral singularities of $L$ are the real zeros of $e^+$ and $e^-$, the integrals over the real axis are divergent in the norm of $L^2(\mathbf{R}_+)$. Now we will investigate the convergence of these integrals in a norm which is weaker than the norm of $L^2(\mathbf{R}_+)$. For this purpose we will use the technique of the regularization of divergent integrals. So we define the following summability factors

$$F^+_{\alpha\beta}(\lambda) = \begin{cases} \dfrac{(\lambda - \lambda_\alpha)^\beta}{\beta!}, & |\lambda - \lambda_\alpha| < \delta, \quad \alpha = 1, ..., v, \\ 0, & |\lambda - \lambda_\alpha| \geqslant \delta, \quad \alpha = 1, ..., v. \end{cases} \tag{4.19}$$

$$F^{-}_{\alpha\beta}(\lambda)=\begin{cases}\dfrac{(\lambda-\lambda_\alpha)^\beta}{\beta!}, & |\lambda-\lambda_\alpha|<\delta, \quad \alpha=v+1, ..., \ell,\\ 0, & |\lambda-\lambda_\alpha|\geqslant\delta, \quad \alpha=v+1, ..., \ell,\end{cases} \tag{4.20}$$

with $\delta>\delta_0$. We can choose $\delta>0$ so small that the $\delta$-neighborhoods of $\lambda_\alpha$, $\alpha=1, ..., v, v+1, ..., \ell$ have no common points and do not contain the point $\lambda=0$. Define the functionals

$$F^{+}\{g_1(\lambda)\}=g_1(\lambda)-\sum_{\alpha=1}^{v}\sum_{\beta=0}^{n_\alpha-1}\left\{\left(\frac{d}{d\lambda}\right)^\beta g_1(\lambda)\right\}_{\lambda=\lambda_\alpha} F^{+}_{\alpha\beta}(\lambda), \tag{4.21}$$

$$F^{-}\{g_2(\lambda)\}=g_2(\lambda)-\sum_{\alpha=v+1}^{l}\sum_{\beta=0}^{n_\alpha-1}\left\{\left(\frac{d}{d\lambda}\right)^\beta g_2(\lambda)\right\}_{\lambda=\lambda_\alpha} F^{-}_{\alpha\beta}(\lambda), \tag{4.22}$$

where $g_1$ and $g_2$ are chosen so that the right-hand side of the above formulae are meaningful. It is evident from (4.19)–(4.20) that $\lambda_1, ..., \lambda_v$ and $\lambda_{v+1}, ..., \lambda_\ell$ are the roots of $F^{+}\{g_1(\lambda)\}=0$ and $F^{-}\{g_2(\lambda)\}=0$ at least of orders $n_1, ..., n_v$ and $n_{v+1}, ..., n_\ell$, respectively. In the following we will use the operators

$$P^{+}f(x)=\frac{1}{2\pi i}\int_{\Gamma_+}\frac{\lambda e_x^{+}(0,\lambda)}{e^{+}(\lambda)}\varphi(x,\lambda)\,\varphi(f,\lambda)\,d\lambda, \tag{4.23}$$

and

$$P^{-}f(x)=\frac{1}{2\pi i}\int_{\Gamma_-}\frac{\lambda e_x^{-}(0,\lambda)}{e^{-}(\lambda)}\varphi(x,\lambda)\,\varphi(f,\lambda)\,d\lambda, \tag{4.24}$$

$$\begin{aligned}I^{+}f(x)=&\frac{1}{2\pi i}\sum_{\alpha=1}^{v}\sum_{\beta=0}^{n_\alpha-1}\left\{\left(\frac{\partial}{\partial\lambda}\right)^\beta[\varphi(x,\lambda)\,\varphi(f,\lambda)]\right\}_{\lambda=\lambda_\alpha}\\ &\times\int_{\Gamma_+}\frac{\lambda e_x^{+}(0,\lambda)}{e^{+}(\lambda)}F^{+}_{\alpha\beta}(\lambda)\,d\lambda\\ &+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\lambda e_x^{+}(0,\lambda)}{e^{+}(\lambda)}F^{+}\{\varphi(x,\lambda)\,\varphi(f,\lambda)\}\,d\lambda,\end{aligned}$$

and

$$\begin{aligned}I^{-}f(x)=&\frac{1}{2\pi i}\sum_{\alpha=v+1}^{\ell}\sum_{\beta=0}^{n_\alpha-1}\left\{\left(\frac{\partial}{\partial\lambda}\right)^\beta[\varphi(x,\lambda)\,\varphi(f,\lambda)]\right\}_{\lambda=\lambda_\alpha}\\ &\times\int_{\Gamma_-}\frac{\lambda e_x^{-}(0,\lambda)}{e^{-}(\lambda)}F^{-}_{\alpha\beta}(\lambda)\,d\lambda\\ &+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\lambda e_x^{-}(0,\lambda)}{e^{-}(\lambda)}F^{-}\{\varphi(x,\lambda)\,\varphi(f,\lambda)\}\,d\lambda.\end{aligned}$$

Since, under the condition (1.4) $e^{+}(x, \lambda)$ and $e^{-}(x, \lambda)$ have an analytic continuation to the half-planes $\operatorname{Im} \lambda > -\varepsilon/2$ and $\operatorname{Im} \lambda < \varepsilon/2$, respectively, we get

$$P^{\pm} f = I^{\pm} f,$$

for $f \in C_0^{\infty}(\mathbf{R}_+)$.

Lemma 4.4. *For each $f \in H_{(n_0+1)}$, there exist a constant $c > 0$ such that*

$$\|I^{\pm} f\|_{-(n_0+1)} \leqslant c \|f\|_{(n_0+1)}, \tag{4.25}$$

*where* $n_0 = \max\{n_1, ..., n_v, n_{v+1}, ..., n_\ell\}$.

Define

$$\Lambda_\alpha^+ = (\lambda_\alpha - \delta, \lambda_\alpha + \delta), \qquad \alpha = 1, ..., v. \tag{4.26}$$

Then $0 \notin \Lambda_\alpha^+$, $\alpha = 1, ..., v$. Using the integral form of remainder in the Taylor formula, we get

$$F^+\{\varphi(x, \lambda)\, \varphi(f, \lambda)\}$$
$$= \begin{cases} \varphi(x, \lambda)\, \varphi(f, \lambda), & \lambda \in \Lambda_0^+ \\ \dfrac{1}{(n_\alpha - 1)!} \displaystyle\int_{\lambda_\alpha}^{\lambda} (\lambda - \xi)^{n_\alpha - 1} \left\{ \left(\frac{\partial}{\partial \xi}\right)^{n_\alpha} [\varphi(x, \xi)\, \varphi(f, \xi)] \right\} d\xi \\ \lambda \in \Lambda_\alpha^+, & \alpha = 1, ..., v, \end{cases} \tag{4.27}$$

where

$$\Lambda_0^+ = \mathbf{R} \Big\backslash \left\{ \bigcup_{\alpha=1}^{v} \Lambda_\alpha^+ \right\}.$$

If we use the notation

$$I_\alpha^+ f(x) = \frac{1}{2\pi i} \int_{\Lambda_\alpha^+} \frac{\lambda e_x^+(0, \lambda)}{e^+(\lambda)} F^+\{\varphi(x, \lambda)\, \varphi(f, \lambda)\}\, d\lambda, \qquad \alpha = 0, 1, ..., v,$$

$$\tilde{I}^+ f(x) = \frac{1}{2\pi i} \sum_{\alpha=1}^{v} \sum_{\beta=0}^{n_\alpha - 1} \left\{ \left(\frac{\partial}{\partial \lambda}\right)^{\beta} [\varphi(x, \lambda)\, \varphi(f, \lambda)] \right\}_{\lambda = \lambda_\alpha}$$
$$\times \int_{\Gamma_+} \frac{\lambda e_x^+(0, \lambda)}{e^+(\lambda)} F_{\alpha\beta}^+(\lambda)\, d\lambda,$$

we obtain

$$I^+ = I_0^+ + I_1^+ + \cdots + I_v^+ + \tilde{I}^+ \tag{4.28}$$

from (4.26) and (4.27). We now show that each of the operators $I_0^+, I_1^+, \ldots, I_v^+$ and $\tilde{I}^+$ is continuous from $H_{(n_0+1)}$ into $H_{-(n_0+1)}$. We start with $\tilde{I}^+$. From (4.19) we obtain the absolute convergence of

$$\int_{\Gamma_+} \frac{\lambda e_x^+(0, \lambda)}{e^+(\lambda)} F_{\alpha\beta}^+(\lambda)\, d\lambda.$$

Using (3.2) and the isomorphism $H_{-n_0} \sim H'_{n_0}$ we see that $\tilde{I}^+$ is continuous from $H_{n_0}$ into $H_{-n_0}$ or from $H_{(n_0+1)}$ into $H_{-(n_0+1)}$. Hence there exists a constant $\tilde{c} > 0$ such that

$$\|\tilde{I}^+ f(x)\|_{-(n_0+1)} \leqslant \tilde{c}\, \|f\|_{(n_0+1)} \tag{4.29}$$

for any $f \in H_{(n_0+1)}$.

Next we want to show the continuity of $I_\alpha^+$, $\alpha = 1, \ldots, v$, from $H_{(n_0+1)}$ into $H_{-(n_0+1)}$. From (4.27) we see that

$$I_\alpha^+ f(x) = \frac{1}{2\pi i (n_\alpha - 1)!} \int_{\Lambda_\alpha^+} \frac{\lambda e_x^+(0, \lambda)}{e^+(\lambda)} \int_{\lambda_\alpha}^{\lambda} (\lambda - \xi)^{n_\alpha - 1}$$
$$\times \left\{ \left( \frac{\partial}{\partial \xi} \right)^{n_\alpha} [\varphi(x, \xi)\, \varphi(f, \xi)] \right\} d\xi\, d\lambda. \tag{4.30}$$

Interchanging the order of integration, we get

$$I_\alpha^+ f(x) = \frac{1}{2\pi i (n_\alpha - 1)!}$$
$$\times \left\{ \int_{\lambda_\alpha}^{\lambda_\alpha + \delta} \int_{\xi}^{\lambda_\alpha + \delta} \left\{ \left( \frac{\partial}{\partial \xi} \right)^{n_\alpha} [\varphi(x, \xi)\, \varphi(f, \xi)] \right\} \right.$$
$$\times (\lambda - \xi)^{n_\alpha - 1} \frac{\lambda e_x^+(0, \lambda)}{e^+(\lambda)}\, d\lambda\, d\xi$$
$$- \int_{\lambda_\alpha - \delta}^{\lambda_\alpha} \int_{\lambda_\alpha - \delta}^{\xi} \left\{ \left( \frac{\partial}{\partial \xi} \right)^{n_\alpha} [\varphi(x, \xi)\, \varphi(f, \xi)] \right\}$$
$$\left. \times (\lambda - \xi)^{n_\alpha - 1} \frac{\lambda e_x^+(0, \lambda)}{e^+(\lambda)}\, d\lambda\, d\xi \right\}.$$

Since $\lambda_\alpha$ is a zero of $e^+$ order $n_\alpha$, there exists a continuous function $e_\alpha^+$ such that $e_\alpha^+(\lambda_\alpha) \neq 0$ and $e^+(\lambda) = (\lambda - \lambda_\alpha)^{n_\alpha} e_\alpha^+(\lambda)$. On the other hand,

$$\left| \int_{\xi}^{\lambda_\alpha + \delta} \frac{\lambda (\lambda - \xi)^{n_\alpha - 1}\, e_x^+(0, \lambda)}{e^+(\lambda)}\, d\lambda \right| \leqslant h_\alpha^+(\xi) [\ln \delta - \ln(\xi - \lambda_\alpha)], \tag{4.31}$$

if $\xi > \lambda_\alpha$, and

$$\left| \int_{\lambda_\alpha - \delta}^{\xi} \frac{\lambda(\lambda - \xi)^{n_\alpha - 1}\, e_x^+(0, \lambda)}{e^+(\lambda)}\, d\lambda \right| \leqslant h_\alpha^{\,-}(\xi)[\ln(\lambda_\alpha - \xi) - \ln \delta], \quad (4.32)$$

if $\xi < \lambda_\alpha$, where

$$h_\alpha^+(\xi) = \max_{\lambda \in [\xi, \lambda_\alpha + \delta]} \left| \frac{\lambda e_x^+(0, \lambda)}{e_\alpha^+(\lambda)} \right|, \qquad h_\alpha^-(\xi) = \max_{\lambda \in [\lambda_\alpha - \delta, \xi]} \left| \frac{\lambda e_x^+(0, \lambda)}{e_\alpha^+(\lambda)} \right|.$$

(4.31) and (4.32) show that $I_\alpha^+$, $\alpha = 1, ..., v$, are integral operators with kernels having logarithmic singularities, i.e., weak singularities. (4.30) can be written as

$$I_\alpha^+ f(x) = \int_{\Lambda_\alpha} \sum_{k=0}^{n_\alpha} b_{k\alpha}(x, \xi) \left\{ \left( \frac{d}{d\xi} \right)^k \varphi(f, \xi) \right\} d\xi.$$

Define

$$B_{k\alpha} = \int_0^\infty \int_{\Lambda_\alpha^+} \left| \frac{b_{k\alpha}(x, \xi)}{(1 + x)^{n_0 + 1}} \right|^2 d\xi\, dx.$$

We see that $B_{k\alpha} < \infty$, by (3.3), (4.31) and (4.32). Since

$$\begin{aligned} \|I_\alpha^+ f\|^2_{-(n_0+1)} &= \int_0^\infty \left| \frac{I_\alpha^+ f(x)}{(1 + x)^{n_0 + 1}} \right|^2 dx \\ &\leqslant \sum_{k=0}^{n_\alpha} \int_0^\infty \int_{\Lambda_\alpha^+} \left| \frac{b_{k\alpha}(x, \xi)}{(1 + x)^{n_0 + 1}} \right|^2 d\xi\, dx \int_{\Lambda_\alpha^+} \left| \left( \frac{d}{d\xi} \right)^k \varphi(f, \xi) \right|^2 d\xi \\ &= \sum_{k=0}^{n_\alpha} B_{k\alpha} \int_{\Lambda_\alpha^+} \left| \left( \frac{d}{d\xi} \right)^k \varphi(f, \xi) \right|^2 d\xi. \end{aligned}$$

Utilizing (4.17) and (4.18) we obtain

$$\|I_\alpha^+ f\|_{-(n_0+1)} \leqslant c_\alpha \|f\|_{n_0} \leqslant c_\alpha \|f\|_{(n_0+1)}, \qquad \alpha = 1, ..., v, \qquad (4.33)$$

where $c_\alpha$ are constants.

Lastly we consider the operator $I_0^+$ which is defined by

$$I_0^+ f(x) = \frac{1}{2\pi i} \int_{-\infty}^\infty \chi_0^+(\lambda) \frac{\lambda e_x^+(0, \lambda)}{e^+(\lambda)} \varphi(x, \lambda)\, \varphi(f, \lambda)\, d\lambda, \qquad (4.34)$$

where $\chi_0^+$ is the characteristic function of the interval $\Lambda_0^+$. From (4.34), similar to the proof of Lemma 4.2, we get

$$\int_0^\infty |I_\alpha^+ f(x)|^2\, dx \leqslant c_0 \int_0^\infty |f(x)|^2\, dx,$$

where $c_0 > 0$ is a constant. Since

$$H_{(n_0+1)} \subsetneqq L^2(\mathbf{R}_+) \subsetneqq H_{-(n_0+1)},$$

we find

$$\|I_0^+ f\|_{-(n_0+1)} \leqslant c_0 \, \|f\|_{(n_0+1)}. \tag{4.35}$$

From (4.28), (4.29), (4.33), and (4.35) we have

$$\|I^+ f\|_{-(n_0+1)} \leqslant c \, \|f\|_{(n_0+1)}.$$

In a similar way it follows that

$$\|I^- f\|_{-(n_0+1)} \leqslant c \, \|f\|_{(n_0+1)}. \quad \blacksquare$$

Then for every $f \in H_{(n_0+1)}$,

$$\begin{aligned} I^+ f(x) = {} & \frac{1}{2\pi i} \sum_{\alpha=1}^{v} \sum_{\beta=0}^{n_\alpha - 1} \left\{ \left( \frac{\partial}{\partial \lambda} \right)^\beta [\varphi(x, \lambda)\, \varphi(f, \lambda)] \right\}_{\lambda = \lambda_\alpha} \\ & \times \int_{\Gamma_+} \frac{\lambda e_x^+(0, \lambda)}{e^+(\lambda)} F_{\alpha\beta}^+(\lambda)\, d\lambda \\ & + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\lambda e_x^+(0, \lambda)}{e^+(\lambda)} F^+ \{\varphi(x, \lambda)\, \varphi(f, \lambda)\}\, d\lambda, \end{aligned} \tag{4.36}$$

and

$$\begin{aligned} I^- f(x) = {} & \frac{1}{2\pi i} \sum_{\alpha=v+1}^{\ell} \sum_{\beta=0}^{n_\alpha - 1} \left\{ \left( \frac{\partial}{\partial \lambda} \right)^\beta [\varphi(x, \lambda)\, \varphi(f, \lambda)] \right\}_{\lambda = \lambda_\alpha} \\ & \times \int_{\Gamma_+} \frac{\lambda e_x^-(0, \lambda)}{e^-(\lambda)} F_{\alpha\beta}^-(\lambda)\, d\lambda \\ & + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\lambda e_x^-(0, \lambda)}{e^-(\lambda)} F^- \{\varphi(x, \lambda)\, \varphi(f, \lambda)\}\, d\lambda. \end{aligned} \tag{4.37}$$

Let $a_\alpha(\lambda)$ denote any function which is defined and differentiable in a neighborhood of $\lambda_\alpha$, and which satisfies the condition

$$\left\{\left(\frac{d}{d\lambda}\right)^{n_\alpha - 1 - \beta} a_\alpha(\lambda)\right\}_{\lambda=\lambda_\alpha}$$
$$= \begin{cases} \dfrac{1}{2\pi i}\dbinom{n_\alpha - 1}{\beta}\displaystyle\int_{\Gamma_+} \frac{\lambda e_x^+(0,\lambda)}{e^+(\lambda)} F^+_{\alpha\beta}(\lambda)\, d\lambda, & \alpha = 1, ..., v \\ -\dfrac{1}{2\pi i}\dbinom{n_\alpha - 1}{\beta}\displaystyle\int_{\Gamma_-} \frac{\lambda e_x^-(0,\lambda)}{e^-(\lambda)} F^-_{\alpha\beta}(\lambda)\, d\lambda, & \alpha = v+1, ..., \ell. \end{cases} \tag{4.38}$$

Then (4.36) and (4.37) can be written as

$$I^+ f(x) = \sum_{\alpha=1}^{v} \left\{\left(\frac{\partial}{\partial\lambda}\right)^{n_\alpha - 1} [a_\alpha(\lambda)\, \varphi(x,\lambda)\, \varphi(f,\lambda)]\right\}_{\lambda=\lambda_\alpha}$$
$$+ \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\lambda e_x^+(0,\lambda)}{e^+(\lambda)} F^+\{\varphi(x,\lambda)\, \varphi(f,\lambda)\}\, d\lambda, \tag{4.39}$$

$$I^- f(x) = -\sum_{\alpha=v+1}^{\ell} \left\{\left(\frac{\partial}{\partial\lambda}\right)^{n_\alpha - 1} [a_\alpha(\lambda)\, \varphi(x,\lambda)\, \varphi(f,\lambda)]\right\}_{\lambda=\lambda_\alpha}$$
$$+ \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\lambda e_x^-(0,\lambda)}{e^-(\lambda)} F^-\{\varphi(x,\lambda)\, \varphi(f,\lambda)\}\, d\lambda. \tag{4.40}$$

We shall also use the following integral operators (see (4.7):

$$Q^+ f(x) = \frac{1}{2\pi i}\int_{\Gamma_+} \frac{e_x^+(0,\lambda)}{e^+(\lambda)}\, \varphi(x,\lambda)\, \varphi(f,\lambda)\, d\lambda, \tag{4.41}$$

$$Q^- f(x) = \frac{1}{2\pi i}\int_{\Gamma_-} \frac{e_x^-(0,\lambda)}{e^-(\lambda)}\, \varphi(x,\lambda)\, \varphi(f,\lambda)\, d\lambda, \tag{4.42}$$

$$J^+ f(x) = \frac{1}{2\pi i}\sum_{\alpha=1}^{v}\sum_{\beta=0}^{n_\alpha - 1} \left\{\left(\frac{\partial}{\partial\lambda}\right)^{\beta} [\varphi(x,\lambda)\, \varphi(f,\lambda)]\right\}_{\lambda=\lambda_\alpha}$$
$$\times \int_{\Gamma_+} \frac{e_x^+(0,\lambda)}{e^+(\lambda)} F^+_{\alpha\beta}(\lambda)\, d\lambda$$
$$+ \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e_x^+(0,\lambda)}{e^+(\lambda)} F^+\{\varphi(x,\lambda)\, \varphi(f,\lambda)\}\, d\lambda,$$

and

$$J^{-}f(x)=\frac{1}{2\pi i}\sum_{\alpha=v+1}^{\ell}\sum_{\beta=0}^{n_\alpha-1}\left\{\left(\frac{\partial}{\partial\lambda}\right)^{\beta}[\varphi(x,\lambda)\,\varphi(f,\lambda)]\right\}_{\lambda=\lambda_\alpha}$$
$$\times\int_{\Gamma_+}\frac{e_x^{-}(0,\lambda)}{e^{-}(\lambda)}F_{\alpha\beta}^{-}(\lambda)\,d\lambda$$
$$+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e_x^{-}(0,\lambda)}{e^{-}(\lambda)}F^{-}\{\varphi(x,\lambda)\,\varphi(f,\lambda)\}\,d\lambda.$$

It is evident that,

$$Q^{\pm}f=J^{\pm}f,$$

for $f\in C_0^\infty(\mathbf{R}_+)$.

Similar to Lemma 4.4, we find

LEMMA 4.5. *For every $f\in H_{(n_0+1)}$ there exist a constant $c>0$ such that*

$$\|J^{\pm}f\|_{-(n_0+1)}\leqslant c\,\|f\|_{(n_0+1)}.$$

*It is evident that, for every $f\in H_{(n_0+1)}$*

$$J^{+}f(x)=\sum_{\alpha=1}^{v}\left\{\left(\frac{\partial}{\partial\lambda}\right)^{n_\alpha-1}[b_\alpha(\lambda)\,\varphi(x,\lambda)\,\varphi(f,\lambda)]\right\}_{\lambda=\lambda_\alpha}$$
$$+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e_x^{+}(0,\lambda)}{e^{+}(\lambda)}F^{+}\{\varphi(x,\lambda)\,\varphi(f,\lambda)\}\,d\lambda, \tag{4.43}$$

$$J^{-}f(x)=-\sum_{\alpha=v+1}^{\ell}\left\{\left(\frac{\partial}{\partial\lambda}\right)^{n_\alpha-1}[b_\alpha(\lambda)\,\varphi(x,\lambda)\,\varphi(f,\lambda)]\right\}_{\lambda=\lambda_\alpha}$$
$$+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e_x^{-}(0,\lambda)}{e^{-}(\lambda)}F^{-}\{\varphi(x,\lambda)\,\varphi(f,\lambda)\}\,d\lambda, \tag{4.44}$$

*where*

$$\left\{\left(\frac{d}{d\lambda}\right)^{n_\alpha-1-\beta}b_\alpha(\lambda)\right\}_{\lambda=\lambda_\alpha}$$
$$=\begin{cases}\dfrac{1}{2\pi i}\dbinom{n_\alpha-1}{\beta}\displaystyle\int_{\Gamma_+}\frac{e_x^{+}(0,\lambda)}{e^{+}(\lambda)}F_{\alpha\beta}^{+}(\lambda)\,d\lambda, & \alpha=1,\ldots,v\\[2ex] -\dfrac{1}{2\pi i}\dbinom{n_\alpha-1}{\beta}\displaystyle\int_{\Gamma_-}\frac{e_x^{-}(0,\lambda)}{e^{-}(\lambda)}F_{\alpha\beta}^{-}(\lambda)\,d\lambda, & \alpha=v+1,\ldots,\ell.\end{cases} \tag{4.45}$$

THEOREM 4.6. *Under the condition* (1.4) *the following two-fold spectral expansion in terms of the principal functions of L holds,*

$$
\begin{aligned}
f(x) = & \sum_{i=1}^{j} \left\{ \left( \frac{\partial}{\partial \lambda} \right)^{m_i^+ - 1} [a_i^+(\lambda)\, \varphi(x, \lambda)\, \varphi(f, \lambda)] \right\}_{\lambda = \lambda_i^+} \\
& + \sum_{i=1}^{k} \left\{ \left( \frac{\partial}{\partial \lambda} \right)^{m_i^- - 1} [a_i^-(\lambda)\, \varphi(x, \lambda)\, \varphi(f, \lambda)] \right\}_{\lambda = \lambda_i^-} \\
& + \sum_{\alpha=1}^{\ell} \left\{ \left( \frac{\partial}{\partial \lambda} \right)^{n_\alpha - 1} [a_\alpha(\lambda)\, \varphi(x, \lambda)\, \varphi(f, \lambda)] \right\}_{\lambda = \lambda_\alpha} \\
& + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\lambda e_x^+(0, \lambda)}{e^+(\lambda)} F^+\{\varphi(x, \lambda)\, \varphi(f, \lambda)\}\, d\lambda \\
& - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\lambda e_x^-(0, \lambda)}{e^-(\lambda)} F^-\{\varphi(x, \lambda)\, \varphi(f, \lambda)\}\, d\lambda,
\end{aligned} \tag{4.46}
$$

$$
\begin{aligned}
0 = & \sum_{i=1}^{j} \left\{ \left( \frac{\partial}{\partial \lambda} \right)^{m_i^+ - 1} [b_i^+(\lambda)\, \varphi(x, \lambda)\, \varphi(f, \lambda)] \right\}_{\lambda = \lambda_i^+} \\
& + \sum_{i=1}^{k} \left\{ \left( \frac{\partial}{\partial \lambda} \right)^{m_i^- - 1} [b_i^-(\lambda)\, \varphi(x, \lambda)\, \varphi(f, \lambda)] \right\}_{\lambda = \lambda_i^-} \\
& + \sum_{\alpha=1}^{\ell} \left\{ \left( \frac{\partial}{\partial \lambda} \right)^{n_\alpha - 1} [b_\alpha(\lambda)\, \varphi(x, \lambda)\, \varphi(f, \lambda)] \right\}_{\lambda = \lambda_\alpha} \\
& + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e_x^+(0, \lambda)}{e^+(\lambda)} F^+\{\varphi(x, \lambda)\, \varphi(f, \lambda)\}\, d\lambda \\
& - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e_x^-(0, \lambda)}{e^-(\lambda)} F^-\{\varphi(x, \lambda)\, \varphi(f, \lambda)\}\, d\lambda\, .
\end{aligned} \tag{4.47}
$$

*for every function $f \in H_{(n_0+1)}$. The integrals in* (4.46) *and* (4.47) *converge in the norm of $H_{-(n_0+1)}$ where $a_i^\pm$, $b_i^\pm$, $F^+$, $F^-$, $a_\alpha$, and $b_\alpha$ defined by* (4.8), (4.9), (4.21), (4.22), (4.38), *and* (4.45), *respectively.*

*Proof.* We obtain (4.46) and (4.47) for $f \in C_0^\infty(\mathbf{R}_+) \subset H_{(n_0+1)}$, by use of (4.6), (4.7), (4.23), (4.24), and (4.39)–(4.44). The convergence of the integrals appearing in (4.46) and (4.47) in the norm of $H_{-(n_0+1)}$ has been given in Lemmas 4.4 and 4.5. Since $C_0^\infty(\mathbf{R}_+)$ is dense in $H_{(n_0+1)}$, the proof is completed. ∎

*Note.* By (3.1), for every $f \in H_{(n_0+1)} \subset L^2(\mathbf{R}_+)$ the integrals

$$
\varphi^{(m)}(f, \lambda_i^+) := \int_0^\infty f(x)\, \varphi^{(m)}(x, \lambda_i^+)\, dx, \qquad m = 0, 1, ..., m_i^+ - 1, \quad i = 1, ..., j, \tag{4.48}
$$

and

$$\varphi^{(m)}(f, \lambda_i^-) := \int_0^\infty f(x)\, \varphi^{(m)}(x, \lambda_i^-)\, dx, \qquad m = 0, 1, ..., m_i^- - 1, \quad i = 1, ..., k, \tag{4.49}$$

converge. The differentiation in the first and second terms on the right-hand side of (4.46) and (4.47) is symbolic: that is, the rule for differentiating a product is first applied as if the derivatives of $\varphi(f, \lambda)$ at the points $\lambda_i^+$ and $\lambda_i^-$ existed, and then these "derivatives" are replaced by the numbers $\varphi^{(m)}(f, \lambda_i^+)$ and $\varphi^{(m)}(f, \lambda_i^-)$ defined by (4.48) and (4.49), respectively. By Lemma 4.3 for every $f \in H_{(n_0+1)}$, the function $\varphi(f, \lambda)$ may be differentiated $(n_0 + 1)$ times with respect to $\lambda$ $(\lambda \in \mathbf{R})$.

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